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ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF ELASTICITY THEORY PROBLEM FOR SHELL OF POSITIVE CURVATURE AND SMALL THICKNESS*

N.A. BAZARENKO

The state of stress and strain of a shell of positive curvature with one edge subjected to the effect of a sufficiently smooth load applied to the endface surface is studied. The case is investigated when the shell thickness is slight. It is proved that the shell state of stress consists of three parts: 1) the internal state of stress that does not possess the property of decay and encloses all domains of the shell body, 2) the slowly decaying state of stress (simple edge effect of shells), and 3) the rapidly decaying state of stress of boundary-layer type. Asymptotic expansions are presented for the components of states of stress and strain of the types 1), 2) and 3). Boundary conditions are formulated for each part of the solution constructed. A system of "two-dimensional" equations of the refined applied theory of shells is obtained on the basis of the solution of a three-dimensional problem of elasticity theory.

1. Initial equations. Let V be the domain of space filled with shell material, $\mathbf{R}$ is the radius-vector of a running point in this domain, S is the shell middle surface, $\mathbf{r} = \mathbf{r}(\alpha, \beta)$ is some orthogonal parametrization of this surface, $\mathbf{n}$ is the normal direction to the surface S. Then the transformation equation $\mathbf{R} = \mathbf{r} + \mathbf{n}t$ yields a semi-orthogonal curvilinear coordinate system x^1, x^2, x^3 in the domain V $(\alpha \equiv x^1,\ \beta \equiv x^2,\ t \equiv x^3)$.

We introduce an orthonormal coordinate basis $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$, where $\mathbf{i}_2, \mathbf{i}_3$ are the tangent directions to the coordinate lines x^2, x^3 while $\mathbf{i}_1 = \mathbf{i}_2 \times \mathbf{i}_3$ is the normal direction to the coordinate surface $x^1 = \text{const}$. We denote the stress tensor components by σ_{ik}^* and the coordinates of the displacement vector by u_k^* in this reference system. We take the elasticity theory equations in a semi-orthogonal coordinate system obtained in /1/, and represented as follows

$$\begin{aligned}
&(u_p^*)_t' = \sigma_{3p} - \zeta_p u_p^* - \delta_p z_0 u_q^* - D^p u_3^* \quad (p \neq q = 1,2) \\
&(u_3^*)_t' = c_4 [(\zeta_1 + \zeta_2) u_3^* - \theta(u_1^*, u_2^*)] - f_4 \sigma_{33} \\
&(\sigma_{3p})_t' = (\zeta_q + 2\zeta_p)\sigma_{3p} + \delta_q z_0 \sigma_{3q} - 2g_1 g_2 u_p^* + \\
&\quad (\delta_q - 1) D^q \theta(u_2,^* - u_1^*) + D^p(\sigma_{33} - c_6\theta^*) + 2(z_0 D^q - \zeta_q D^p) u_3^* \\
&(\sigma_{33})_t' = 2(\zeta_1 + \zeta_2)(\sigma_{33} - f_6\theta^*) - \theta(\sigma_{31}, \sigma_{32}) - 4g_1 g_2 u_3^* + \\
&\quad 2(\zeta_2 D^1 - z_0 D^2 + z_0 z_2 + z_1\zeta_1) u_1^* + 2(\zeta_1 D^2 - z_0 D^1 + z_0 z_1 + z_2\zeta_2) u_2^* \\
&\sigma_{pp} = 2(D^p u_p^* + z_q u_q^* - \zeta_p u_3^* + f_5\theta^*) \\
&\sigma_{12} = (D^1 - z_1) u_2^* + (D^2 - z_2) u_1^* - 2z_0 u_3^* \\
&\theta^* = \sigma_{33} + 2[\theta(u_1^*, u_2^*) - (\zeta_1 + \zeta_2) u_3^*], \quad \theta(w_1, w_2) \equiv \\
&\quad (D^1 + z_1) w_1 + (D^2 + z_2) w_2 \\
&D^1 = \sqrt{g_{22}/g}\, \partial/\partial\alpha - (g_{12}/\sqrt{g g_{22}})\, \partial/\partial\beta, \quad D^2 = (1/\sqrt{g_{22}})\, \partial/\partial\beta, \\
&\delta_p = 1 + (-1)^p \\
&g = \det \| g_{ik} \|, \ g_p = k_p/(1 - k_p t), \ \sigma_{ik} = \sigma_{ik}^*/\mu, \ 2\varkappa = (1 - \nu)^{-1} \\
&c_{rs} = (1 + r)\varkappa - 3 + s/2, \ c_{0s} \equiv f_s, \ c_{1s} \equiv c_s \ (r, s = 0, 1, 2, \ldots 9)
\end{aligned} \tag{1.1}$$

as the initial relations.

Here g_{ik} are the metric tensor components, k_1 and k_2 are the principal curvatures of the surface S, μ is the shear modulus, and ν is the Poisson's ratio. The functions z_p and ζ_p satisfy the Gauss–Peterson–Codazzi equations

$$\begin{aligned}
&D^q\zeta_p - D^p z_0 = 2z_0 z_p + z_q(\zeta_q - \zeta_p), \quad g_1 g_2 = -z_1^2 - z_2^2 - D^1 z_1 - D^2 z_2 \\
&(\zeta_p)_t' = \zeta_p^2 + (2\delta_q - 1) z_0^2, \quad (z_p)_t' = z_p\zeta_2 + {}^1/_2(\delta_q D^q z_0 - \delta_p D^p \zeta_1) \\
&(z_0)_t' = 2z_0\zeta_2, \ \zeta_1 + \zeta_2 = g_1 + g_2, \quad z_0^2 = \zeta_1\zeta_2 - g_1 g_2 \ (p \neq q = 1, 2)
\end{aligned} \tag{1.2}$$

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The relationships

$$\begin{aligned}
&(D^p)_t' = \zeta_p D^p + \delta_q z_0 D^q, \quad (D^p)|_{t=0} = A_p \partial/\partial x^p \equiv \partial_p^* \\
&(z_0)|_{t=0} = MA_1A_2 = m \\
&(D^p + z_p) D^q \equiv (D^q + z_q) D^p, \quad (\zeta_p)|_{t=0} = A_p^2 B_p = k_{pp} \\
&(z_p)|_{t=0} = -\partial_p^* \ln A_q = k_p^* \\
&k_{11} \equiv k_\alpha,\ k_{22} \equiv k_\beta,\ A_1 = 1/\sqrt{E},\ A_2 = 1/\sqrt{G},\ B_1 = L,\ B_2 = N \\
&H = k_\alpha + k_\beta = k_1 + k_2,\ m^2 = k_\alpha k_\beta - k_1 k_2 \quad (p \neq q = 1, 2)
\end{aligned} \tag{1.3}$$

hold together with (1.2).

Here k_{pp} and $(-1)^p k_q^*$ are, respectively, the normal and geodesic curvatures of the coordinate line $x^q =$ const on the middle surface, $(-1)^q m$ is the geodesic torsion of the surface S in the direction of this same line, and E, G, L, M, N are coefficients of the first and second quadratic forms.

Integrating (1.1) by using power series in the coordinate t, and using the relationships (1.2) and (1.3) here, as well as the symbolic writing of A.I. Lur'e /2,3/, we obtain

$$u_k^* = u_k + \sum_{s=1}^{\infty} t^s (A_{k,s}^j u_j + B_{k,s}\sigma_j), \quad \sigma_{pq} = \sum_{s=0}^{\infty} t^s (A_{pq,s}^j u_j + B_{pq,s}^j \sigma_j) \tag{1.4}$$

$$\sigma_{3k} = \sigma_k + \sum_{s=1}^{\infty} t^s (A_{3k,s}^j u_j + B_{3k,s}^j \sigma_j) \quad (p, q = 1, 2) \tag{1.5}$$

Here $A_{k,s}^j, \ldots, B_{ik,s}^j$ are known differential operators /1/, $u_j = u_j^*(\alpha, \beta, 0)$ and $\sigma_j = \sigma_{3j}(\alpha, \beta, 0)$ are the displacement and stress on the middle surface for $t = 0$.

It can be shown that all the coefficients $A_{ik,s}^j u_j$ are expressed in terms of the quantities $\varepsilon, \omega \equiv \{\varepsilon_1, \varepsilon_2, \omega\}$ and $\varkappa, \tau \equiv \{\varkappa_1, \varkappa_2, \tau\}$ are respectively the components of the tangential and bending strains of the middle surface /4/

$$\begin{aligned}
&A_{pp,0}^j u_j \equiv t_p = 2(c_6\varepsilon_p + c_4\varepsilon_q), \quad A_{12,0}^j u_j = \omega \quad (p \neq q = 1, 2) \\
&A_{pp,1}^j u_j = m_p + (2\delta_q - 1) m\omega + k_{pp} t_p \\
&A_{12,1}^j u_j = 2\tau + H\omega + m(t_2 + 2\varepsilon_2) \\
&A_{3p,1}^j u_j \equiv L_p(t, \omega) = k_p^* t_q - (\partial_p^* + k_p^*) t_p - (\partial_q^* + 2k_q^*)\omega \\
&m_p = 2(c_6\varkappa_p + c_4\varkappa_q) \\
&A_{33,1}^j u_j \equiv L_3(t, \omega) = -k_\alpha t_1 - k_\beta t_2 - 2m\omega, \ m_1 + m_2 = 4c_5 m^* \\
&A_{ik,2}^j u_j \equiv \Pi_{ik}(\varkappa, \tau) + \Pi_{ik}^*(\varepsilon, \omega), \\
&\Pi_{pp} = c_4 k_{qq} m^* + k_{pp} m_p + (4\delta_q - 2) m\tau \\
&\Pi_{12} = H\tau + m(c_6 m^* + m_2 - m_1), \quad \Pi_{3p} = -c_6 \partial_p^* m^*, \quad \Pi_{33} = -c_6 H m^*
\end{aligned} \tag{1.6}$$

The coordinate system α, β, t is used to study the internal, thin-shell state of stress varying smoothly in the domain V. Another part of the state of stress, localized in the boundary-layer zone and decaying exponentially with distance from the shell edge, is investigated in a system of local semi-geodesic coordinates n, s, t. To this end, orthogonal semi-geodesic parametrization $\mathbf{r} = \mathbf{r}(n, s)$ is introduced on the middle surface, whose single egde is determined by a regular closed line Γ, so that the family of coordinate lines $s =$ const will consist of geodesics perpendicular to Γ. The line Γ is here determined by the equation $n = 0$, and the coordinate s is its natural parameter.

Furthermore, to indicate in which coordinate system the components σ_{ik}, u_k, etc. have been obtained, we rename them by replacing the superscipts 1 and 2 by appropriate letters.

2. Internal state of stress and strain. Let Γ_1 and Γ_2 be parts of the shell surface given by $\zeta = \pm 1$ and $n = 0$, respectively ($\zeta = t/h$, where h is half the shell thickness). Let us extract the homogeneous solutions out of (1.4) and (1.5), i.e., solutions which keep the boundary Γ_1 stress-free

$$\sigma_{3i} = 0 \text{ as } \zeta = \pm 1 \ (i = 1, 2, 3) \tag{2.1}$$

and permit satisfaction of the boundary conditions on the endface surface Γ_2

$$\sigma_{nn}^* = q_1^*, \quad \sigma_{ns}^* = q_2^* \quad \sigma_{n3}^* = q_3^* \text{ as } n = 0 \tag{2.2}$$

where $q_i^* = \mu q_i(s, \zeta)$ are coordinates of the external force intensity vector.

Taking account of (1.5), we write the system (2.1) thus:

$$\sigma_i + \sum_{s=1}^{\infty} h^{2s} (A_{3i,2s}^j u_j + B_{3i,2s}^j \sigma_j) = 0 \tag{2.3}$$

$$\sum_{s=0}^{\infty} h^{2s}(A^j_{3i,\,2s+1}u_j + B^j_{3i,\,2s+1}\sigma_j) = 0 \quad (i = 1, 2, 3) \tag{2.4}$$

We shall seek the undamped solution of the singularly-perturbed system presented above, as $h \to 0$. In this case the operators $A^j_{3i,\,s}$ and $B^j_{3i,\,s}$ applied to the functions u_j and σ_j do not change their order of smallness in h, while the stresses σ_j admit of asymptotic expansion:

$$\sigma_j = \sigma_{j,\,0} + h^2\sigma_{j,\,1} + h^4\sigma_{j,\,2} + \ldots \; (j = 1, 2, 3) \tag{2.5}$$

Substituting (2.5) into (2.3) and equating the expression for h^{2k} $(k = 0, 1, 2\ldots)$ to zero, we obtain a system of recursion equations in the unknowns $\sigma_{j,\,r}$ $(r = 0, 1, 2, \ldots)$. We hence find

$$\begin{aligned} &\sigma_j = -h^2 A^i_{3j,\,2}u_i - h^4 C^i_{j,\,422}u_i + \ldots \quad (j = 1, 2, 3) \\ &C^i_{j,\,mrl} = A^i_{3j,\,m} - B^k_{3j,\,r}A^i_{3k,\,l} \quad (m, r, \quad l = 0, 1, 2, \ldots, 9) \end{aligned} \tag{2.6}$$

Eliminating the stress σ_j from (1.4), (1.5) and (2.4) by using (2.6), we obtain

$$u_k^* = u_k + h\zeta A^i_{k,\,1}u_i + h^2\zeta^2 A^i_{k,\,2}u_i + \ldots \tag{2.7}$$

$$\begin{aligned} &\sigma_{pq} = A^i_{pq,\,0}u_i + h\zeta A^i_{pq,\,1}u_i + h^2(\zeta^2 A^i_{pq,\,2} - B^k_{pq,\,0}A^i_{3k,\,2})\,u_i + \ldots \\ &\sigma_{3k} = h^2(\zeta^2 - 1)(A^i_{3k,\,2}u_i + h\zeta A^i_{3k,\,3}u_i + \ldots) \quad (p, q = 1, 2) \\ &A^k_{3j,\,1}u_k + h^2 C^k_{j,\,312}u_k + h^4(C^k_{j,\,532} - B^i_{3j,\,1}C^k_{i,\,422})\,u_k + \ldots = 0 \\ &C^k_{j,\,312}u_k \equiv \Lambda_j(\varkappa, \tau) + \Lambda_j^*(\varepsilon, \omega) \\ &\Lambda_3 = -\frac{2}{3}\,[c_{54}(f_6 H^2 - k_1k_2) + c_6\nabla]\,m^* \\ &\Lambda_p = \frac{1}{3}\,\{[(c_{54}H - c_4k_{pp})\,\partial_p^* - c_4m\partial_q^*]\,m^* - c_6c_{54}\partial_p^*(Hm^*) + \\ &\quad \partial_q^*[(k_{pp} - k_{qq})\,\tau + m(\varkappa_q - \varkappa_p)]\} \\ &\nabla \equiv (\partial_1^* + k_1^*)\,\partial_1^* + (\partial_2^* + k_2^*)\,\partial_2^*, \quad m^* = \varkappa_1 + \varkappa_2 \\ &(p \neq q = 1, 2; \quad j = 1, 2, 3) \end{aligned} \tag{2.8}$$

Furthermore, by appending the strain continuity equation to (2.8) /4/

$$\Omega_j(\varkappa, \tau) \equiv L_j(m, 2\tau) - 2\Pi_{3j}(\varkappa, \tau) = R_j(\varepsilon, \omega) \quad (j = 1, 2, 3) \tag{2.9}$$

and selecting the quantities of the strain components ε, ω and $\varkappa, \tau$ as unknowns, we will seek the undamped solution of the system obtained in the form of asymptotic expansions

$$\varepsilon_j = \sum_{r=0} h^{r/2}\varepsilon_{j,\,r}, \quad \varkappa_j = h^{-2}\sum_{r=0} h^{r/2}\varkappa_{j,\,r} \quad (\varepsilon_3 \equiv \omega, \varkappa_3 \equiv \tau, j = 1, 2, 3) \tag{2.10}$$

Now, substituting (2.10) into (2.8), (2.9) and equating the expression for $h^{r/2}$ to zero, we obtain a system of recurrent equations in the functions $\varepsilon_{j,\,r}$ and $\varkappa_{j,\,r}$

$$\begin{aligned} &L_{j,\,r}(t, \omega) = -\Lambda_{j,\,r}(\varkappa, \tau), \; \Omega_{j,\,r}(\varkappa, \tau) = 0 \quad (j = 1, 2, 3) \\ &\Omega_{j,\,l}(\varkappa, \tau) = R_{j,\,r}(\varepsilon, \omega) \quad \text{etc.} \quad (l = r + 4, \; r = 0, 1, 2, 3) \end{aligned} \tag{2.11}$$

For instance, writing $L_{p,\,r}(t, \omega)$ or $t_{p,\,r}$ is decoded thus

$$\begin{aligned} &L_{p,\,r}(t, \omega) \equiv k_p^* t_{q,\,r} - (\partial_p^* + k_p^*)\,t_{p,\,r} - (\partial_q^* + 2k_q^*)\,\omega_r \\ &t_{p,\,r} \equiv 2(c_6\varepsilon_{p,\,r} + c_4\varepsilon_{q,\,r}) \end{aligned}$$

Determining the functions $\varepsilon_{j,\,r}$ and $\varkappa_{j,\,r}$ from (2.11), and then substituting (2.10) into (2.7), we find the asymptotic expansion of the components of the shell internal state of stress and strain. Taking account of (1.6), we obtain

$$\begin{aligned} &\sigma_{pp} = h^{-1}\zeta m_{p,\,0} + h^{-1/2}\zeta m_{p,\,1} + (\zeta m_{p,\,2} + t_{p,\,0} + \zeta^2\Pi_{pp,\,0} - c_4\Pi_{33,\,0}) + \ldots \\ &\sigma_{12} = h^{-1}2\zeta\tau_0 + h^{-1/2}2\zeta\tau_1 + (2\zeta\tau_2 + \omega_0 + \zeta^2\Pi_{12,\,0}) + \ldots \\ &\sigma_{3k} = (\zeta^2 - 1)(\Pi_{3k,\,0} + h^{1/2}\Pi_{3k,\,1} + \ldots), \quad u_k = h^{-2}u_{k,\,0} + h^{-3/2}u_{k,\,1} + \ldots \\ &u_p^* = h^{-2}u_{p,\,0} + h^{-3/2}u_{p,\,1} + h^{-1}[u_{p,\,2} - \zeta(\partial_p^* u_{3,\,0} + k_{pp}u_{p,\,0} + \\ &\quad \delta_p m u_{q,\,0})] + \ldots \\ &u_3^* = h^{-2}u_{3,\,0} + h^{-3/2}u_{3,\,1} + h^{-1}u_{3,\,2} + \ldots \quad (p \neq q = 1, 2) \end{aligned} \tag{2.12}$$

Let us note that the problem of determining the middle surface displacements by means of given strain components is solved in quadratures /4,5/, here the quantities $u_{k,\,r}$ are found in terms of the functions $\varkappa_{j,\,r}$ and $\varepsilon_{j,\,r-4}$.

Let us introduce the specific forces T_p, S_{qp}, N_p and the moments G_p, H_{qp} originating on the shell coordinate sections $x^p = \text{const}$ $(p \neq q = 1, 2)$

$$A_q \int_{-h}^{h} \sigma_{(p)} \sqrt{g_{qq}}\, dt = T_p \mathbf{i}_{p0} + S_{qp}\mathbf{i}_{q0} - N_p \mathbf{i}_3 = \mu h \sum_{r=0} h^{r/2} (T_{p,r}\mathbf{i}_{p0} + S_{qp,r}\mathbf{i}_{q0} - N_{p,r}\mathbf{i}_3) \tag{2.13}$$

$$A_q \int_{-h}^{h} (\sigma_{(p)} \times \mathbf{i}_3)\, t \sqrt{g_{qq}}\, dt = (-1)^q (H_{qp}\mathbf{i}_{p0} + G_p \mathbf{i}_{q0}) = (-1)^q \mu h \sum_{r=0} h^{r/2} (H_{qp,r}\mathbf{i}_{p0} + G_{p,r}\mathbf{i}_{q0})$$

Here $\sigma_{(p)}$ is the stress vector on the surface $x^p = \text{const}$, and $\mathbf{i}_{p0} = (\mathbf{i}_p)|_{t=0}$. Substituting (2.12) into (2.13), we obtain

$$T_{p,r} = 2t_{p,r} + \frac{2}{3} c_{54} m_r^* (c_6 H - k_{qq}) \tag{2.14}$$

$$S_{12,r} + S_{21,r} = 4\omega_r + \frac{4}{3} c_{54} m m_r^*$$

$$H_{12,r} = H_{21,r} = \frac{4}{3}\tau_r, \qquad G_{p,r} = -\frac{2}{3} m_{p,r},$$

$$N_{p,r} = -\frac{8}{3} f_6 \partial_p^* m_r^* = (f_6/c_5)\, \partial_p^* (G_{1,r} + G_{2,r})$$

$$S_{21,r} - S_{12,r} = (k_\alpha - k_\beta) H_{21,r} + m (G_{1,r} - G_{2,r}) \tag{2.15}$$

$$(p \neq q = 1, 2;\quad r = 0, 1, 2, 3)$$

Eliminating the quantities $t_{p,r}$, ω_r and $m_{p,r}$, τ_r from (2.11) by using (2.14) we obtain

$$(\partial_p^* + k_p^*) T_{p,r} - k_p^* T_{q,r} + (\partial_q^* + k_q^*) S_{pq,r} + k_q^* S_{qp,r} + k_{pp} N_{p,r} + m N_{q,r} = 0 \tag{2.16}$$

$$k_{11} T_{1,r} + k_{22} T_{2,r} + m (S_{12,r} + S_{21,r}) - (\partial_1^* + k_1^*) N_{1,r} - (\partial_2^* + k_2^*) N_{2,r} = 0$$

$$k_p^* G_{q,r} - (\partial_p^* + k_p^*) G_{p,r} + (\partial_q^* + 2k_q^*) H_{21,r} + N_{p,r} = 0$$

$$(p \neq q = 1, 2;\quad r = 0, 1, 2, 3)$$

$$2m H_{21,r} - k_{11} G_{1,r} - k_{22} G_{2,r} + (f_6/c_5) H (G_{1,r} + G_{2,r}) = 0$$

The first six equations from (2.15) and (2.16) agree in form with the equilibrium equations of general shell theory /4/. Hence, (2.14) should be considered as an equation of state. It can be shown that the internal state of stress described by the relationships (2.12)—(2.16) is the sum of membrane and purely couple-stress states.

3. Simple edge effect. By using the system of local coordinates n, s, t we seek the solution of (2.1) localized in the boundary layer, for which the following asymptotic relationships are characteristic

$$\partial_p u_j = O(u_j/h^{\nu_p}),\ \partial_p \sigma_j = O(\sigma_j/h^{\nu_p})\ (p = 1, 2;\ \nu_1 = 1/2,\ \nu_2 = 0) \tag{3.1}$$

$$(\partial_1 \equiv \partial/\partial n,\ \partial_2 \equiv \partial/\partial s)$$

Estimating the orders of the differential operators applied to the displacements u_i in the expansions (2.6), it can be established that the expansions (2.6)—(2.8) hold also for the solution possessing the property (3.1). As is known /6/, the system (2.1) reduces to one constitutive equation by using the stress function in the case of circular cylindrical and spherical shells. For shells of arbitrary shape an asymptotic analog of the stress function, the function φ_3, is successfully obtained. We set

$$u_p = M_{3p}\varphi_3 + \varphi_p,\ u_3 = M_{33}\varphi_3\ (p \neq q = 1, 2) \tag{3.2}$$

$$M_{3p} = M_{3p}^* + X_{p,1} d_1^2 + X_{p,2} d_1 d_2 + \ldots + X_{p,6}$$

$$d_2^m d_1^k = d_1^k d_2^m \equiv A_2^m \partial_2^m \partial_1^k$$

$$M_{33} \equiv M_{33}^* + Y_1 d_1^3 + Y_2 d_1^2 d_2 + \ldots + Y_{10},$$

$$M_{33}^* = \varkappa (d_1^4 + 2d_1^2 d_2^2 + d_2^4)$$

$$M_{3p}^* = (\varkappa H - k_{qq}/2)\, d_p^3 - c_4 m\, d_q d_p^2 + (c_{25} k_{pp} - \varkappa k_{qq})\, d_q^2 d_p + c_6 m\, d_q^3$$

Here $X_{p,r}$ and Y_m $(r = 1, 2, \ldots, 6;\ m = 1, 2, \ldots, 10)$ are functions to be determined, M_{3j}^* are cofactors to the elements a_{3j} of the determinant $\det\|a_{ij}\|$ $(a_{ij} \equiv A_{3i,1}^j)$ in which the symbols ∂_1 and ∂_2 are considered as numbers.

Substituting (3.2) into (2.8), we obtain

$$A_{3j,1}^{k} M_{3k} \varphi_3 + A_{3j,1}^{p} \varphi_p + h^2 (C_{j,312}^{k} M_{3k} \varphi_3 + C_{j,312}^{p} \varphi_p) + \ldots = 0 \quad (j = 1, 2, 3) \tag{3.3}$$

(summation over repeated superscript p from 1 to 2).

Because of the selection of M_{3k}* the operators $A_{3j,1}^{k} M_{3k}$ have the form

$$A_{3j,1}^{k} M_{3k} = \sum_{l,r=0}^{4} a_{lr}^{(j)} d_1^l d_2^r \qquad (l + r \leqslant 4, \quad j = 1, 2, 3)$$

Now, having the functions $X_{p,r}$ satisfying the conditions

$$a_{40}^{(p)} = a_{31}^{(p)} = a_{22}^{(p)} = a_{30}^{(p)} = a_{21}^{(p)} = a_{20}^{(p)} = 0 \qquad (p = 1, 2)$$

at our disposal, we use the arbitrariness of the functions Y_m, so that all the remaining coefficients $a_{lr}^{(p)}$ $(l + r \leqslant 4)$ would vanish for circular cylindrical and spherical shells.

We hence find the quantities $X_{p,r}, Y_m$ and $a_{lr}^{(j)}$ some of which are presented below

$$\begin{aligned}
&X_{1,1} = k_n^* (c_{25} k_n - \varkappa k_s) - (c_3 k_s + \varkappa H)_n', \qquad Y_1 = 2\varkappa k_n^*, \\
&Y_2 = 2\varkappa (A_2)_s' \\
&X_{2,1} = A_2 (c_6 H + f_5 k_s)_s' - 5\varkappa m_n', \qquad A_{33,1}^{k} M_{3k} = 2c_5 \{k_s^2 d_1^4 - \\
&\quad 4k_s m\, d_1^3 d_2 + (2k_n k_s + 4m^2)\, d_1^2 d_2^2 - 4k_n m\, d_1 d_2^3 + k_n^2 d_2^4 + \\
&\quad 2\, [k_n^* k_s^2 - A_2 (m k_s)_s']\, d_1^3 + 2\, [(A_2 k_n k_s + 2A_2 m^2)_s' + (m k_s)_n' + \\
&\quad 2k_s (m_n' - A_2 H_s')]\, d_1^2 d_2 + \ldots\}
\end{aligned} \tag{3.4}$$

Furthermore, following /7/ and taking account of (3.4), we expand the coefficients in (3.3) in a power series in the coordinate n and we stretch the scale

$$\begin{aligned}
&n = h^{1/2}\xi, \qquad \partial_1 = h^{-1/2}\partial_{10}, \qquad \partial_{10} \equiv \partial/\partial\xi, \qquad A_2 = 1 - h^{1/2}\xi k_g + \ldots \\
&k_{s0} = k_s|_{n=0}, \qquad m_0 = m|_{n=0}, \qquad k_g = k_n^*|_{n=0}, \ \text{etc.}
\end{aligned} \tag{3.5}$$

Finally, by seeking the unknowns φ_j in the form of the expansions

$$\varphi_p = h^{1/2} \sum_{r=0} h^{r/2} \varphi_{pr}, \qquad \varphi_3 = h \sum_{r=0} h^{r/2} \varphi_{3r} \qquad (p = 1, 2) \tag{3.6}$$

and substituting (3.6) into (3.2), (3.3) and (2.7), we obtain

$$\begin{aligned}
&h^{-1/2} [-\partial_{10}^2 \varphi_{10} + (\varkappa f_5 H_0 - k_{s0} c_5/3)\, \partial_{10}^7 \varphi_{30}] + \ldots = 0 \\
&h^{-1/2} \left[-\partial_{10}^2 \varphi_{20} + \frac{1}{3} \varkappa c_2 m_0 \partial_{10}^7 \varphi_{30}\right] + \ldots = 0 \qquad (c^2 = 3(1 - \nu^2)) \\
&h^{-1} (c^2 k_{s0}^2 \partial_{10}^4 + \partial_{10}^8)\, \varphi_{30} + h^{-1/2} \{(c^2 k_{s0}^2 \partial_{10}^4 + \partial_{10}^8)\, \varphi_{31} + 4k_g \partial_{10}^7 \varphi_{30} + \\
&\quad 2c^2\, [\xi k_{s0} (\partial k_s/\partial n)_0\, \partial_{10}^4 - 2k_{s0} m_0 \partial_{10}^3 \partial_2 + \\
&\quad (k_g k_{s0}^2 - (m_0 k_{s0})_s')\, \partial_{10}^3]\, \varphi_{30}\} + \ldots = 0
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\sigma_{nn} = -h^{-1} 4\varkappa^2 \zeta \partial_{10}^6 \varphi_{30} - h^{-1/2} 2\, [c_6 \zeta (\varkappa \partial_{10}^6 \varphi_{31} + c_{25} k_g \partial_{10}^5 \varphi_{30}) + \\
&\quad c_5 k_g k_{s0} \partial_{10}^3 \varphi_{30}] + \ldots \\
&\sigma_{ss} = -h^{-1} 2\, (c_5 k_{s0} \partial_{10}^4 + \varkappa \zeta c_4 \partial_{10}^6)\, \varphi_{30} + h^{-1/2} \{-c_6 \zeta (c_4 \partial_{10}^6 \varphi_{31} + \\
&\quad c_{52} k_g \partial_{10}^5 \varphi_{30}) + 2c_5\, [2m_0 \partial_{10}^3 \partial_2 - \xi (\partial k_s/\partial n)_0\, \partial_{10}^4 - \\
&\quad (k_g k_{n0} - 2 (\partial k_s/\partial n)_0)\, \partial_{10}^3]\, \varphi_{30} - 2c_5 k_{s0} \partial_{10}^4 \varphi_{31}\} + \ldots \\
&\sigma_{33} = (\zeta^2 - 1) \left\{c_6 \left[(\varkappa H_0 - k_{s0}/2)\, \partial_{10}^6 - \frac{1}{3} \varkappa \zeta \partial_{10}^8\right] \varphi_{30} + \ldots\right\} \\
&\sigma_{n3} = (\zeta^2 - 1)\, [h^{-1/2} 2\varkappa^2 \partial_{10}^7 \varphi_{30} + 2\varkappa^2 (\partial_{10}^7 \varphi_{31} + 3k_g \partial_{10}^6 \varphi_{30}) + \ldots] \\
&\sigma_{ns} = h^{-1/2} 2\, [c_5 \partial_{10}^3 \partial_2 (k_{s0} \varphi_{30}) - \varkappa \zeta \partial_{10}^5 \partial_2 \varphi_{30}] + \ldots \\
&\sigma_{s3} = (\zeta^2 - 1)\, (2\varkappa^2 \partial_{10}^6 \partial_2 \varphi_{30} + \ldots), \\
&u_n^* = h^{-1/2} [(\varkappa H_0 - k_{s0}/2)\, \partial_{10}^3 - \varkappa \zeta \partial_{10}^5]\, \varphi_{30} + \ldots \\
&u_s^* = h^{-1/2} c_6 m_0 \partial_{10}^3 \varphi_{30} + \ldots, \\
&u_3^* = h^{-1} \varkappa \partial_{10}^4 \varphi_{30} + h^{-1/2} \varkappa (\partial_{10}^4 \varphi_{31} + 2k_g \partial_{10}^3 \varphi_{30}) + \ldots
\end{aligned} \tag{3.8}$$

For small h we find from (3.7)

$$\begin{aligned}
&\varphi_{30} = \psi_0, \quad \varphi_{31} = \psi_1 - \xi\, [(f\psi_0)_s' + (E_1 + E_2)\, \psi_0] + \xi^2 E_1 \partial_{10} \psi_0, \\
&f = m_0/k_{s0}
\end{aligned} \tag{3.9}$$

$$\psi_k = \frac{3}{64} \varkappa^{-2} \gamma^{-6}\, [(M_k - Q_k/\gamma) \cos \gamma\xi + M_k \sin \gamma\xi] \exp (\gamma\xi),$$

$$\gamma = \sqrt{ck_{s0}/2}$$
$$E_1 = (4k_{s0})^{-1}[(\partial k_s/\partial n)_0 + f(\partial k_s/\partial s)_0],$$
$$E_2 = (2k_{s0})^{-1}[4(\partial k_s/\partial n)_0 + k_g k_{n0}]$$

where $M_k = M_k(s)$, $Q_k = Q_k(s)$ are functions determined from the boundary conditions on Γ_2.

4. Boundary-layer type of state of stress and strain. We shall seek the decaying solution of the system (2.1) that is characterized by the asymptotic relations (3.1) for $\nu_1 = 1$, $\nu_2 = 0$. To this end, we expand the coefficients of (1.1) in a power series in n and ζ and stretch the scale. We finally obtain

$$(v_1^*)_\zeta' - \sigma_{13} + \partial_{11}v_3^* + hF_1 + \ldots = 0, \quad (4.1)$$
$$(v_2^*)_\zeta' - \sigma_{23} + hF_2 + \ldots = 0$$
$$(v_3^*)_\zeta' + f_4\sigma_{33} + c_4\partial_{11}v_1^* + hF_3 + \ldots = 0,$$
$$(\sigma_{33})_\zeta' + \partial_{11}\sigma_{13} + hF_6 + \ldots = 0$$
$$(\sigma_{13})_\zeta' + c_4\partial_{11}\sigma_{33} + 4\varkappa\partial_{11}^2 v_1^* + hF_4 + \ldots = 0,$$
$$(\sigma_{23})_\zeta' + \partial_{11}^2 v_2^* + hF_5 + \ldots = 0$$
$$\sigma_{12} = \partial_{11}v_2^* + hF_7 + \ldots, \qquad \sigma_{11} = c_4\sigma_{33} + 4\varkappa\partial_{11}v_1^* + hF_8 + \ldots \quad (4.2)$$
$$\sigma_{22} = c_4\sigma_{33} + 2c_4\partial_{11}v_1^* + hF_9 + \ldots, \qquad v_j^* = u_j^*/h \quad (j = 1, 2, 3)$$
$$n = h\rho, \quad \partial_1 = h^{-1}\partial_{11}, \quad \partial_{11} \equiv \partial/\partial\rho, \quad F_1 = k_{n0}(v_1^* + \zeta\partial_{11}v_3^*) \text{ and so on.}$$

Seeking the unknowns v_j^* and σ_{3j} in the form of the series

$$v_j^* = \sum_{r=0}^{\infty} h^r v_{j,r}^*, \qquad \sigma_{3j} = \sum_{r=0}^{\infty} h^r \sigma_{3j,r} \qquad (j = 1, 2, 3)$$

and integrating (4.1) under the initial conditions

$$v_{j,0}^*|_{\zeta=0} = v_j \equiv u_j/h, \qquad \sigma_{3j,0}|_{\zeta=0} = \sigma_j, \qquad v_{j,k}^*|_{\zeta=0} = \sigma_{3j,k}|_{\zeta=0} = 0$$
$$(k = 1, 2, \ldots)$$

we successively find $v_{j,k}^*$ and $\sigma_{3j,k}$ $(k = 0, 1, 2, \ldots)$

$$u_n^*/h = {}^1/_2(\varkappa z\cos z - f_2\sin z)\sigma_1/\partial_{11} - {}^1/_2\varkappa z\sin z\sigma_3/\partial_{11} + (\cos z - \varkappa z\sin z)v_1 + (f_4\sin z - \varkappa z\cos z)v_3 + hv_{1,1}^* + \ldots \quad (4.3)$$
$$u_s^*/h = \sin z\sigma_2/\partial_{11} + \cos zv_2 + hv_{2,1}^* + \ldots$$
$$u_3^*/h = -{}^1/_2\varkappa z\sin z\sigma_1/\partial_{11} - {}^1/_2(\varkappa z\cos z + f_2\sin z)\sigma_3/\partial_{11} - (f_4\sin z + \varkappa z\cos z)v_1 + (\cos z + \varkappa z\sin z)v_3 + hv_{3,1}^* + \ldots$$
$$\sigma_{n3} = (\cos z - \varkappa z\sin z)\sigma_1 - (f_4\sin z + \varkappa z\cos z)\sigma_3 - 2\varkappa(\sin z + z\cos z)\partial_{11}v_1 + 2\varkappa z\sin z\partial_{11}v_3 + h\sigma_{n3,1} + \ldots \quad (4.4)$$
$$\sigma_{s3} = \cos z\sigma_2 - \partial_{11}\sin zv_2 + h\sigma_{s3,1} + \ldots$$
$$\sigma_{33} = (f_4\sin z - \varkappa z\cos z)\sigma_1 + (\cos z + \varkappa z\sin z)\sigma_3 + 2\varkappa z\sin z\partial_{11}v_1 + 2\varkappa(z\cos z - \sin z)\partial_{11}v_3 + h\sigma_{33,1} + \ldots$$

Taking account of (4.3) and (4.4), we find from (4.2)

$$\sigma_{nn} = (f_8\sin z + \varkappa z\cos z)\sigma_1 + (c_4\cos z - \varkappa z\sin z)\sigma_3 + 2\varkappa(2\cos z - z\sin z)\partial_{11}v_1 - 2\varkappa(\sin z + z\cos z)\partial_{11}v_3 + h\sigma_{nn,1} + \ldots \quad (4.5)$$
$$\sigma_{ss} = c_4(\sin z\sigma_1 + \cos z\sigma_3 + 2\partial_{11}\cos z\, v_1 - 2\partial_{11}\sin zv_3) + h\sigma_{ss,1} + \ldots$$
$$\sigma_{ns} = \sin z\,\sigma_2 + \partial_{11}\cos zv_2 + h\sigma_{ns,1} + \ldots \qquad (z \equiv \zeta\partial_{11})$$

Moreover, let the unknowns v_j and σ_j be determined by the asymptotic expansions

$$v_j = \sum_{k=0} h^{k/2}v_{j,k}, \qquad \sigma_r = \sum_{k=0} h^{k/2}\sigma_{r,k}, \qquad \sigma_2 = h^{-1}\sum_{k=0} h^{k/2}\sigma_{2,k} \quad (4.6)$$
$$(r = 1, 3;\ j = 1, 2, 3)$$

We note that since the stress σ_{nj} of the simple edge effect is described by a power series in $h^{1/2}$ and are used in satisfying the boundary conditions on Γ_2, then the asymptotic expansions (2.10) and (4.6) should also have the same configuration.

Now, taking account of (4.4) and (4.6), we obtain the principal boundary-layer equations from (2.1)

$$
\begin{aligned}
&(\cos\partial_{11} - \varkappa\partial_{11}\sin\partial_{11})\,\sigma_{1,l} + 2\varkappa\partial_{11}^2\sin\partial_{11}v_{3,l} = 0\\
&(f_4\sin\partial_{11} - \varkappa\partial_{11}\cos\partial_{11})\,\sigma_{1,l} + 2\varkappa(\partial_{11}^2\cos\partial_{11} - \partial_{11}\sin\partial_{11})\,v_{3,l} = 0\\
&(f_4\sin\partial_{11} + \varkappa\partial_{11}\cos\partial_{11})\,\sigma_{3,l} + 2\varkappa(\partial_{11}\sin\partial_{11} + \partial_{11}^2\cos\partial_{11})\,v_{1,l} = 0\\
&(\cos\partial_{11} + \varkappa\partial_{11}\sin\partial_{11})\,\sigma_{3,l} + 2\varkappa\partial_{11}^2\sin\partial_{11}v_{1,l} = 0\\
&\cos\partial_{11}\sigma_{2,l} = 0, \qquad \partial_{11}\sin\partial_{11}v_{2,l} = 0 \qquad (l = 0, 1)
\end{aligned}
\tag{4.7}
$$

Determining the unknowns $\sigma_{j,l}$ and $v_{j,l}$ from (4.7), and then substituting (4.6) into (4.3)—(4.5), we obtain asymptotic expansions of the boundary-layer type components of the state of stress and strain

$$
\begin{aligned}
&\sigma_{ij} = \sum_{l=0}\sum_{k=1}^{\infty} h^{l/2}\sigma_{ij,l}^{(k)}, \qquad \sigma_{r2} = h^{-1}\sum_{l=0}\sum_{k=1}^{\infty} h^{l/2}\sigma_{r2,l}^{(k)} \quad (r = 1, 3;\ ij \neq r2)\\
&u_r^* = h\sum_{l=0}\sum_{k=1}^{\infty} h^{l/2}u_{r,l}^{(k)}, \qquad u_2^* = \sum_{l=0}\sum_{k=1}^{\infty} h^{l/2}u_{2,l}^{(k)}\\
&\sigma_{nn,l}^{(k)} = c_6\,[C_{k,l}^*(\Psi_k^{(2)} - 2\cos z_k\zeta) + D_{k,l}^*(\Psi_k^{(1)} + 2\sin\theta_k\zeta)] +\\
&\quad 2x_k^{-1}(m_0\cos x_k\zeta - \partial_2\sin x_k\zeta)\,A_{k,l}^*, \qquad \sigma_{ns,l}^{(k)} = A_{k,l}^*\sin x_k\zeta\\
&\sigma_{ns,2+l}^{(k)} = A_{k,2+l}^*\sin x_k\zeta + B_{k,l}^*\cos y_k\zeta + {}^1\!/_2A_{k,l}^*\{\cos x_k\zeta[k_{n0}x_k(\zeta^2-1) +\\
&\quad 3k_{s0}/x_k] + \sin x_k\zeta\,[(k_{s0} + 2k_{n0})\zeta - k_g(\rho + 3/x_k)]\},\\
&\sigma_{s3,l}^{(k)} = A_{k,l}^*\cos x_k\zeta\\
&\sigma_{n3,l}^{(k)} = c_6\,(C_{k,l}^*\Psi_k^{(1)} - D_{k,l}^*\Psi_k^{(3)}) - x_k^{-1}\partial_2\cos x_k\zeta A_{k,l}^*,\\
&A_{k,l}^* = A_{k,l}\exp(x_k\rho)\\
&\sigma_{33,l}^{(k)} = -c_6\,(C_{k,l}^*\Psi_k^{(2)} + D_{k,l}^*\Psi_k^{(4)}) + 2m_0x_k^{-1}\cos x_k\zeta A_{k,l}^*,\\
&B_{k,l}^* = B_{k,l}\exp(y_k\rho)\\
&\sigma_{s3,2+l}^{(k)} = A_{k,2+l}^*\cos x_k\zeta - B_{k,l}^*\sin y_k\zeta + {}^1\!/_2A_{k,l}^*\{\cos x_k\zeta\,[(k_{s0}+2k_{n0})\zeta -\\
&\quad \rho k_g] + x_kk_{n0}(1-\zeta^2)\sin x_k\zeta\},\\
&\sigma_{ss,l}^{(k)} = 2c_4\,(D_{k,l}^*\sin\theta_k\zeta - C_{k,l}^*\cos z_k\zeta) +\\
&\quad 2x_k^{-1}(2\nu m_0\cos x_k\zeta + \partial_2\sin x_k\zeta)\,A_{k,l}^*, \qquad u_{s,l}^{(k)} = x_k^{-1}A_{k,l}^*\sin x_k\zeta\\
&u_{s,2+l}^{(k)} = x_k^{-1}A_{k,2+l}^*\sin x_k\zeta + y_k^{-1}B_{k,l}^*\cos y_k\zeta +\\
&\quad {}^1\!/_2A_{k,l}^*\{k_{n0}(\zeta^2-1)\cos x_k\zeta - \rho k_g\sin x_k\zeta/x_k +\\
&\quad k_{s0}(\zeta\sin x_k\zeta/x_k + 3\cos x_k\zeta/x_k^2)\}\\
&u_{3,l}^{(k)} = C_{k,l}^*z_k^{-1}(\varkappa\Psi_k^{(1)} - \sin z_k\zeta) - D_{k,l}^*\theta_k^{-1}(\varkappa\Psi_k^{(3)} + \cos\theta_k\zeta) +\\
&\quad (1-2\nu)\,m_0x_k^{-2}\sin x_k\zeta A_{k,l}^*\\
&u_{n,l}^{(k)} = C_{k,l}^*z_k^{-1}(\varkappa\Psi_k^{(2)} - \cos z_k\zeta) + D_{k,l}^*\theta_k^{-1}(\varkappa\Psi_k^{(4)} + \sin\theta_k\zeta) +\\
&\quad x_k^{-2}\,[(1-2\nu)\,m_0\cos x_k\zeta - \partial_2\sin x_k\zeta]\,A_{k,l}^* \qquad (l = 0, 1)\\
&\Psi_k^{(2)} = z_k\zeta\sin z_k\zeta + \sin^2 z_k\cos z_k\zeta, \qquad z_k\Psi_k^{(1)} = \partial\Psi_k^{(2)}/\partial\zeta,\\
&C_{k,l}^* = C_{k,l}\exp(z_k\rho)/\cos z_k\\
&\Psi_k^{(4)} = \theta_k\zeta\cos\theta_k\zeta - \cos^2\theta_k\sin\theta_k\zeta, \qquad \theta_k\Psi_k^{(3)} = -\,\partial\Psi_k^{(4)}/\partial\zeta,\\
&D_{k,l}^* = D_{k,l}\exp(\theta_k\rho)/\sin\theta_k
\end{aligned}
\tag{4.8}
$$

Here the numbers x_k, y_k, z_k, θ_k are nonzero roots of the appropriate equations

$$\cos x = 0, \quad \sin y = 0, \ \sin 2z = -2z, \quad \sin 2\theta = 2\theta \quad (x_k > 0, \ldots, \operatorname{Re}\theta_k > 0)$$

and the functions $A_{k,l} = A_{k,l}(s), \ldots, D_{k,l} = D_{k,l}(s)$ are to be determined from the boundary conditions on Γ_2.

In a first approximation the relations (4.8) agree with the homogeneous solutions obtained in slab theory /8,9/, and for $n = 0$ the following hold

$$
\begin{aligned}
&\int_{-1}^{1}\sigma_{nn,l}^{(k)}(C, D)\,d\zeta = 0, \qquad \int_{-1}^{1}\sigma_{ns,l}^{(k)}(A)\,d\zeta = 0, \qquad \int_{-1}^{1}\sigma_{ns,2+l}^{(k)}(B)\,d\zeta = 0\\
&\int_{-1}^{1}\sigma_{n3,l}^{(k)}(C, D)\,d\zeta = 0, \qquad \int_{-1}^{1}\zeta\sigma_{nn,l}^{(k)}(C, D)\,d\zeta = 0 \ \ (l = 0, 1;\ k = 1, 2, \ldots)
\end{aligned}
\tag{4.9}
$$

Here, for instance, only that part of the expression $\sigma_{nn,l}^{(k)}$ which is proportional to the functions $C_{k,l}$ and $D_{k,l}$ is denoted by $\sigma_{nn,l}^{(k)}(C, D)$.

It follows from (4.9) that the system of stresses originating on the boundary Γ_2 is self-equilibrated over the shell thickness in a first approximation, and therefore, the state of stre⁻⁻ of boundary-layer type is a Saint-Venant edge effect.

5. Satisfaction of the boundary conditions. We examine the problem of complete reduction of the system of external stresses from the endface surface Γ_2. We seek the general solution of this problem in the form of a sum of the internal state of stress and strain (1), the simple shell edge effect (2) and the Saint-Venant type boundary layers (3)

$$u_i^* = u_i^{*(1)} + u_i^{*(2)} + u_i^{*(3)}, \qquad \sigma_{ik} = \sigma_{ik}^{(1)} + \sigma_{ik}^{(2)} + \sigma_{ik}^{(3)} \tag{5.1}$$

The stresses and displacements in (5.1) are given by (2.12), (3.8) and (4.8). By virtue of (5.1) the boundary conditions (2.2) become

$$\begin{gathered}\sigma_{np}\,|_{n=0} = h^{-1}\sigma^0_{np,0} + h^{-1/2}\sigma^0_{np,1} + \sigma^0_{np,2} + \ldots = q_{p,0} + h^{1/2} q_{p,1} + \ldots \\ \sigma_{n3}\,|_{n=0} = h^{-1/2}\sigma^0_{n3,1} + \sigma^0_{n3,2} + \ldots = q_{3,0} + h^{1/2} q_{3,1} + \ldots \\ \left(q_k = \sum_{r=0} h^{r/2} q_{k,r}\right)\end{gathered} \tag{5.2}$$

Hence, equating the expressions for h^{-1} and $h^{-1/2}$ to zero, we find

$$\begin{gathered}Q_0 = 0, \quad M_0 = G_{n,0}\,|_{n=0}, \quad A_{k,0} = -3x_k^{-2}\sin x_k H_{sn,0}\,|_{n=0} \\ M_1 = \{G_{n,1} + 6\partial_2 (fG_{n,0}/\gamma) + [6E_2 - 24E_1 - (2+\nu)\,k_g]\,G_{n,0}/\gamma\}\,|_{n=0} \\ A_{k,1} = 3x_k^{-2}\sin x_k [-H_{sn,1} + (\nu-1)\partial_2 (G_{n,0}/\gamma)]\,|_{n=0}\ (k = 1, 2, \ldots \infty)\end{gathered} \tag{5.3}$$

Moreover, taking account of (4.9) and integrating (5.2) with respect to ζ, we obtain a system of boundary conditions for solutions of the type (1) and (2)

$$\begin{gathered}\int_{-1}^{1} (\sigma^0_{nj,\,2+r} - q_{j,\,r})\,d\zeta = 0, \qquad \int_{-1}^{1} \zeta\,(\sigma^0_{nn,\,2+r} - q_{1,\,r})\,d\zeta = 0 \\ (j = 1, 2, 3;\ r = 0, 1, 2, \ldots)\end{gathered} \tag{5.4}$$

Hence, for $r = 0$ it follows

$$\begin{gathered}\{T'_{n,0} - N'_{n,0}k_g/k_{s0} - (m_s' k_g/k_{s0} - m_0^2 + 2k_g f\partial_2 + \partial_2^2)(G_{n,0}/k_{s0})\}\,|_{n=0} = \\ T_0^* - N_0^* k_g/k_{s0} \quad (T'_{n,0} = T_{n,0} - mH_{sn,0}, \quad N'_{n,0} = N_{n,0} - \partial_2 H_{sn,0}) \\ \{S'_{sn,0} + m_0 G_{n,0} + \partial_2 [N'_{n,0}/k_{s0} + (m_s'/k_{s0} + 2f\partial_2)(G_{n,0}/k_{s0})] - \\ k_g \partial_2 (G_{n,0}/k_{s0})\}\,|_{n=0} = S_0^* + \partial_2 (N_0^*/k_{s0}) \quad (S'_{sn,0} = S_{sn,0} - k_s H_{sn,0}) \\ Q_1 = [(7E_2 - 35E_1 - 3k_g)\,G_{n,0} + 7\partial_2 (fG_{n,0}) + N'_{n,0}]\,|_{n=0} - N_0^* \\ T_0^* = \int_{-1}^{1} q_{1,0}\,d\zeta, \quad S_0^* = \int_{-1}^{1} q_{2,0}\,d\zeta, \quad N_0^* = -\int_{-1}^{1} q_{3,0}\,d\zeta\end{gathered} \tag{5.5}$$

Here $T'_{n,0}$, $S'_{sn,0}$, $N'_{n,0}$ are reduced edge forces /4/. For determination of functions $B_{k,0}$, $C_{k,0}$, $D_{k,0}$ in (4.8), we use the Lagrange principle of possible displacements. Since homogeneous solutions satisfy the equilibrium equations and boundary conditions on Γ_1, then the variational equation takes the form

$$\begin{gathered}\int_{\Gamma_2} (\sigma_{nn}\delta u_n^* + \sigma_{ns}\delta u_s^* + \sigma_{n3}\delta u_3^*)\,d\sigma = \int_{\Gamma_2} (q_1\delta u_n^* + q_2\delta u_s^* + q_3\delta u_3^*)\,d\sigma \\ d\sigma = [(1 - k_{s0}t)^2 + m_0^2 t^2]^{1/2}\,dt\,ds\end{gathered} \tag{5.6}$$

Varying the functions $B_{k,0}$ ($k = 1, 2, \ldots \infty$), we obtain from (5.6)

$$B_{k,0} = \int_{-1}^{1} q_{2,0}\cos y_k\zeta\,d\zeta + 6y_k^{-2}\cos y_k \left[\left(k_{n0} - \frac{3}{4}k_{s0}\right) H_{sn,0} + m_0 (G_{s,0} - \nu G_{n,0})\,c_{44}/c_{34}\right]\Big|_{n=0} \tag{5.7}$$

As is seen from (4.8), the stresses $\sigma^{(k)}_{nn,l}$ and $\sigma^{(k)}_{n3,l}$ ($l = 0, 1$) are proportional to the coefficient $\varkappa$ ($0.5 \leqslant \varkappa \leqslant 1$). By varying the function $C_{k,0}$ and $D_{k,0}$ and obtaining a system of linear algebraic equations from (5.6) for $\varkappa = 0.5$, this permits construction of an appropriate system for an arbitrary value of $\varkappa$. We have

$$\begin{gathered}C_{k,0}\left(1 - \frac{2}{3}\sin^2 z_k\right) z_k^{-1} + 8\sum_{\substack{m=1 \\ m\neq k}}^{\infty} C_{m,0} z_k z_m (\sin^2 z_k - \sin^2 z_m)(z_k - z_m)^{-3}(z_k + z_m)^{-2} = \\ (2\varkappa z_k \cos z_k)^{-1}\left\{\int_{-1}^{1} [q_{1,0}\,({}^1/{}_2\Psi_k^{(2)} - \cos z_k\zeta) + \right.\end{gathered} \tag{5.8}$$

$$q_{3,0}(^1/_2\Psi_k^{(1)} - \sin z_k\zeta)]\,d\zeta + 3\,[k_{s0}(\nu G_{n,0} - G_{s,0})\,c_4/c_{34} + 8m_0H_{sn,0}]\,z_k^{-3}\sin z_k\}\big|_{n=0}$$

$$D_{k,0}\left(1-\frac{2}{3}\cos^2\theta_k\right)\theta_k^{-1} + 8\sum_{\substack{m=1\\ m\neq k}}^{\infty} D_{m,0}\theta_k\theta_m(\cos^2\theta_k - \cos^2\theta_m)(\theta_k-\theta_m)^{-3}(\theta_k+\theta_m)^{-2} =$$

$$(2\varkappa\theta_k\sin\theta_k)^{-1}\left\{\int_{-1}^{1}[q_{1,0}(^1/_2\Psi_k^{(4)} + \sin\theta_k\zeta) -\right.$$

$$q_{3,0}(^1/_2\Psi_k^{(3)} + \cos\theta_k\zeta)]\,d\zeta - 6N_0{}^*\cos^{-3}\theta_k -$$

$$\left.12\partial_2H_{sn,0}\theta_k\cos\theta_k\sum_{r=1}^{\infty}x_r^{-1}(x_r{}^2-\theta_k{}^2)^{-2}\right\}\Bigg|_{n=0} \qquad (k=1,2,\ldots\infty)$$

The systems (5.8) encountered in slab theory are always solvable, and the method of truncation /8,9/ is used for their solution.

Let us indicate the sequence of seeking solutions of the type (1), (2) and (3). When the forces $T_0{}^*, S_0{}^*, N_0{}^*$ on Γ_2 are not simultaneously zero, we find firstly the quantities $T_{p,0}, S_{qp,0}, N_{p,0}, G_{p,0}, H_{21,0}$ characterizing the internal state of stress by integrating the differential equations (2.16) in combination with the boundary conditions (5.5). Then by using the boundary conditions (5.3) and (5.5) as well as the infinite systems (5.7) and (5.8), we determine the functions M_0, Q_1 and the functions $A_{k,0}, B_{k,0}, C_{k,0}, D_{k,0}$ $(k=1,2,\ldots\infty)$ comprising the arbitrariness of the solutions of the simple edge effect equations and the boundary-layer equations, respectively. If $T_0{}^* = S_0{}^* = N_0{}^* = 0$ on the shell edge, then as follows from (2.16), (5.3) and (5.5), the quantities $T_{p,0}, S_{qp,0}, N_{p,0}, G_{p,0}, H_{21,0}, M_0, Q_1$ must be set equal to zero and the computation must be started with the boundary-layer, i.e., with the solution of the systems (5.7), (5.8).

It is expedient to consider the relations (2.12)—(2.16) and (3.7)—(3.9) resulting from the solution of a three-dimensional problem of elasticity theory together with the boundary conditions (5.3)—(5.5) as a system of "two-dimensional" equations of the refined applied theory intended to reduce the stress from the endface surface Γ_2. By assuming a boundary-layer type solution (4.8) here, the boundary conditions on Γ_2 can be satisfied more exactly than in the integral sense. We note that the results of this paper are valid even for shells of zero and negative curvature if only the contour Γ bounding the middle surface of these shells has a non-asymptotic direction throughout.

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